Equation Sheet

1 Newtons second law and equations of kinematics

The net force $\mathbf{F}$ on a particle of mass m is given by

$$\sum_i \mathbf{F}_i = \mathbf{F} = m\mathbf{a} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}. \tag{1}$$

For a system of particles

$$\mathbf{F} = m\mathbf{a}_G = m\dot{\mathbf{v}}_G. \tag{2}$$

2 Linear and curvilinear motion

2.1 Motion along a path

$$\begin{aligned} &s, \\ &\dot{s} = v. \\ &\ddot{s} = \dot{v} = a, \\ &a\mathrm{d}s = v\mathrm{d}v. \end{aligned} \tag{3}$$

2.2 Angular motion relations

$$\begin{aligned} &\theta, \\ &\dot{\theta} = \omega. \\ &\ddot{\theta} = \dot{\omega} = \alpha, \\ &\alpha\,\mathrm{d}\theta = \omega\,\mathrm{d}\omega. \end{aligned} \tag{4}$$

2.3 Projectile motion

$$
\begin{aligned}
a_x &= 0, \quad a_y = -g, \\
v_x &= (v_x)_o, \quad v_y = (v_y)_o - gt, \\
x &= x_o + (v_x)_o t, \quad y = y_o + (v_y)_o t - g\frac{t^2}{2}, \\
v_y^2 &= (v_y)_o^2 - 2g\left(y - y_o\right).
\end{aligned} \tag{5}
$$

3 Vector relations

3.1 Cartesian

$$
\begin{aligned}
\mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \\
\dot{\mathbf{r}} &= \mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}, \\
\ddot{\mathbf{r}} &= \dot{\mathbf{v}} = \mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}.
\end{aligned} \tag{6}
$$

3.2 Normal and tangential

$$
\begin{aligned}
\mathbf{v} &= v\mathbf{e}_t, \\
\dot{\mathbf{v}} &= \mathbf{a} = \dot{v}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n.
\end{aligned} \tag{7}
$$

3.3 Polar coordinates

$$
\begin{aligned}
\mathbf{r} &= r\mathbf{e}_r, \\
\dot{\mathbf{r}} &= \mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{k}, \\
\ddot{\mathbf{r}} &= \dot{\mathbf{v}} = \mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\mathbf{k}.
\end{aligned} \tag{8}
$$

Note that α is often used to represent $\ddot{\theta}$ and ω is substituted for $\dot{\theta}$.

4 Relative velocities and accelerations (vector addition)

Position of A relative to B is denoted $\mathbf{r}_{A/B}$

Similar vector addition rules apply for velocity and acceleration.

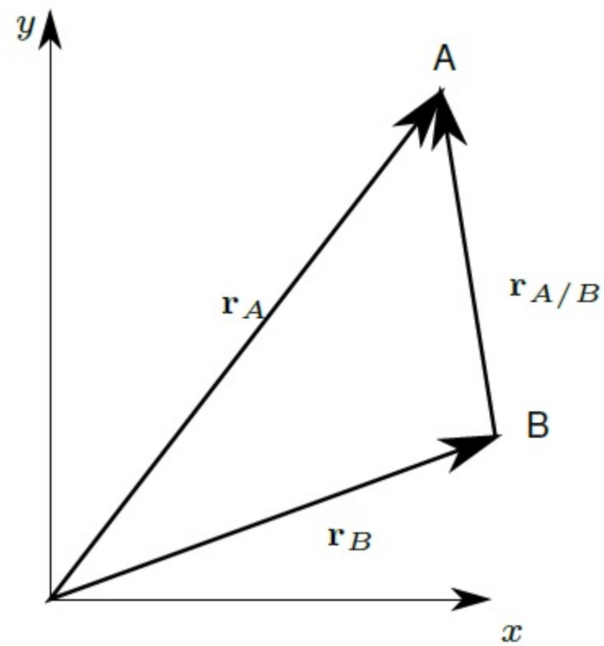

Fig. 8: Rule of vector addition.

$$\begin{aligned}\mathbf{r}_A &= \mathbf{r}_B + \mathbf{r}_{A/B},\\ \mathbf{v}_A &= \mathbf{v}_B + \mathbf{v}_{A/B},\\ \mathbf{a}_A &= \mathbf{a}_B + \mathbf{a}_{A/B}.\end{aligned} \tag{9}$$

Relative to a point B the position, velocity, and acceleration of a point A can be determined from

$$\begin{aligned}\mathbf{v}_{A/B} &= \boldsymbol{\omega}_{A/B} \times \mathbf{r}_{A/B},\\ \mathbf{a}_{A/B} &= \left(\mathbf{a}_{A/B}\right)_n + \left(\mathbf{a}_{A/B}\right)_t,\\ \left(\mathbf{a}_{A/B}\right)_t &= \boldsymbol{\alpha}_{A/B} \times \mathbf{r}_{A/B},\\ \left(\mathbf{a}_{A/B}\right)_n &= \boldsymbol{\omega}_{A/B} \times \left(\boldsymbol{\omega}_{A/B} \times \mathbf{r}_{A/B}\right).\end{aligned} \tag{10}$$

5 Work

$$\mathrm{d}U = \mathbf{F} \cdot \mathrm{d}\mathbf{r} = F \mathrm{d}r \cos(\gamma) = F_t \mathrm{d}s. \tag{11}$$

$$U_{1-2} = \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_1^2 F_t \mathrm{d}s \tag{12}$$

For an external force at angle γ with linear translation in direction $\mathrm{d}x$

$$U_{1-2} = \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_1^2 F\cos(\gamma)\mathrm{d}x \tag{13}$$

Work associated with a spring (a negative value indicates work done on the spring)

$$U_{1-2} = -\frac{1}{2}k\left(x_2^2 - x_1^2\right). \tag{14}$$

Work associated with gravity for small relative distances

$$U_{1-2} = -mg(y_2 - y_1). \tag{15}$$

Work associated with friction

$$U_{1-2} = -\mu F_N \left|x_2 - x_1\right|. \tag{16}$$

5.1 Work-energy equation

$$T_1 + U_{1-2} = T_2. \tag{17}$$

5.2 Work-energy equation(potential)

$$T_1 + V_1 + U'_{1-2} = T_2 + V_2, \tag{18}$$

$$V = V_g + V_e, \tag{19}$$

or

$$U'_{1-2} = \Delta T + \Delta V. \tag{20}$$

5.3 Potential energy

Stored.

Gravitational

$$V_g = mgh. \tag{21}$$

Elastic (linear spring)

$$V_e = \frac{1}{2}kx^2. \tag{22}$$

For a system of particles

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}\sum_i m_i \left|\dot{\boldsymbol{\rho}}_i\right|^2. \tag{23}$$

For rigid body, plane motion

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2, \tag{24}$$

$$= \frac{1}{2}I_O\omega^2. \tag{25}$$

Equation 25 is for rotation about a fixed axis.
Force on an object in the field of scalar potential

$$\mathbf{F} = -\Delta V(x, y, z), \tag{26}$$

$$\Delta = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{27}$$

6 Momentum and impulse equations

6.1 Distance to center of mass

$$\mathbf{r}_G = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}, \tag{28}$$

or

$$m\mathbf{r}_G = \sum m_i \mathbf{r}_i. \tag{29}$$

6.2 Linear momentum and impulse

$$\mathbf{G}_1 + \int_{t_1}^{t_2} \sum \mathbf{F} \mathrm{d}t = \mathbf{G}_2. \tag{30}$$

For a system of particles

$$\mathbf{G} = m\mathbf{v}_G. \tag{31}$$

and

$$\sum \mathbf{F} = \dot{\mathbf{G}}. \tag{32}$$

6.3 Moment of momentum and angular impulse

Angular momentum and moment of momentum are the same thing.

For a system of particles measured from a fixed point 'O'

$$\mathbf{H}_O = \sum \mathbf{r}_i \times m_i \mathbf{v}_i, \tag{33}$$

$$\sum \mathbf{M}_O = \sum \mathbf{r} \times \mathbf{F} = \dot{\mathbf{H}}_O. \tag{34}$$

6.4 Angular impulse equation

$$\mathbf{H}_{O_1} + \int_{t_1}^{t_2} \sum \mathbf{M}_O \mathrm{d}t + \mathbf{H}_{O_2}. \tag{35}$$

6.5 About center of mass

$$\mathbf{H}_G = \sum_i \boldsymbol{\rho}_i \times m_i \mathbf{v}_i. \tag{36}$$

$$\sum \mathbf{M}_G = \dot{\mathbf{H}}_G. \tag{37}$$

6.6 About arbitrary point P

$$\mathbf{H}_P = \mathbf{H}_G + \boldsymbol{\rho}_{G/P} \times m \mathbf{v}_G. \tag{38}$$

$$\sum \mathbf{M}_P = \dot{\mathbf{H}}_G + \boldsymbol{\rho}_{G/P} \times m \mathbf{a}_G. \tag{39}$$

Relative moment of momentum about P is

$$(\mathbf{H}_P)_{\text{rel}} = \mathbf{H}_G + \boldsymbol{\rho}_{G/P} \times m \mathbf{v}_{P/G}. \tag{40}$$

$$\sum \mathbf{M}_P = (\dot{\mathbf{H}}_P)_{\text{rel}} + \boldsymbol{\rho}_{G/P} \times m \mathbf{a}_P. \tag{41}$$

or, equivalently

$$\sum \mathbf{M}_P = \dot{\mathbf{H}}_G + \boldsymbol{\rho}_{G/P} \times m \mathbf{a}_G. \tag{42}$$

6.7 Rigid body moments of momentum

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In addition to $\mathbf{F} = m\mathbf{a}_G$,

$$\sum \mathbf{M}_G = \dot{\mathbf{H}}_G. \tag{43}$$

$$\sum \mathbf{M}_P = \dot{\mathbf{H}}_G + \boldsymbol{\rho}_{G/P} \times m\mathbf{a}_G. \tag{44}$$

$$\sum \mathbf{M}_P = I_P\boldsymbol{\alpha} + \boldsymbol{\rho}_{G/P} \times m\mathbf{a}_P. \tag{45}$$

or, equivalently

$$\sum \mathbf{M}_P = I_G\boldsymbol{\alpha} + \boldsymbol{\rho}_{G/P} \times m\mathbf{a}_G. \tag{46}$$

6.8 Planar rigid bodies

All moments and rotation axes share a common unit vector direction.

$$\sum M_G = I_G\alpha, \tag{47}$$

$$\sum M_O = I_O\alpha, \tag{48}$$

$$\sum M_P = I_G\alpha + ma_Gd, \tag{49}$$

$$\sum M_P = I_P\alpha + ma_Pd. \tag{50}$$

If P is not accelerating P becomes O

$$d = \frac{|\boldsymbol{\rho} \times \mathbf{a}|}{a}. \tag{51}$$

$$H_o = I_G\omega + mv_Gd = I_O\omega. \tag{52}$$

7 Second moment of mass about a centroidal axis

$$I_G = \int^{\text{Vol.}} r^2 \mathrm{d}m. \tag{53}$$

Parallel axis theorem

$$I_P = I_G + mL^2. \tag{54}$$

P refers to a point that is fixed relative to the rigid body.

8 Impact

8.1 Coefficient of restitution

Linear impact

$$e = \frac{v_2' - v_1'}{v_1 - v_2}. \tag{55}$$

8.2 Coefficient of restitution

Oblique impact.

$$e = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n}. \tag{56}$$

9 Rotation about a fixed axis

9.1 In vector form

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R}, \tag{57}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{R}, \tag{58}$$

$$= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) + \boldsymbol{\alpha} \times \mathbf{R} = \mathbf{a}_n + \mathbf{a}_t. \tag{59}$$

9.2 Scalar form

Constant R.

$$v = R\omega, \tag{60}$$

$$a_n = R\omega^2 = \frac{v^2}{R} = v\omega, \tag{61}$$

$$a_t = R\alpha. \tag{62}$$

10 Motion relative to rotating axes

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel}, \tag{63}$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v}_{rel} + \mathbf{a}_{rel}. \tag{64}$$

11- Laplace Table

	$f(t)$	$F(s)$
1	Unit Impulse $\delta(t)$	1
2	Unit Step $u(t)$	$\frac{1}{s}$
3	Unit Ramp $r(t)$	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!}, n = 1, 2, 3 \ldots$	$\frac{1}{s^n}$
5	$t^n, n = 1, 2, 3 \ldots$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{t^{n-1}}{(n-1)!}e^{-at}, n = 1, 2, 3 \ldots$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at}, n = 1, 2, 3 \ldots$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
12	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})]$	$\frac{1}{s(s+a)(s+b)}$
18	$\frac{1}{a^2}(1-e^{-at}-ate^{-at}$	$\frac{1}{s(s+a)^2}$

19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_{nt}}\sin(\omega_n\sqrt{1-\zeta^2}t)$	$\frac{{\omega_n}^2}{s^2+2\zeta\omega_n s+{\omega_n}^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_{nt}}\sin\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right)$ $\phi = \tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta})$	$\frac{s}{s^2+2\zeta\omega_n s+{\omega_n}^2}$
24	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_{nt}}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)$ $\phi = \tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta})$	$\frac{{\omega_n}^2}{s(s^2+2\zeta\omega_n s+{\omega_n}^2)}$
25	$1-\cos(\omega t)$	$\frac{\omega^2}{s(s^2+\omega^2)}$
26	$\omega t - \sin(wt)$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
27	$\sin(\omega t) - \omega t\cos(\omega t)$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
28	$\frac{1}{2\omega}t\sin(\omega t)$	$\frac{s}{(s^2+\omega^2)^2}$
29	$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
30	$\frac{1}{{\omega_2}^2-{\omega_1}^2}[\cos(\omega_1 t)-\cos(\omega_2 t)], {\omega_1}^2 \neq {\omega_2}^2$	$\frac{s}{(s^2+{\omega_1}^2)(s^2+{\omega_2}^2)}$
31	$\frac{1}{2\omega}[\sin(\omega t)+\omega t\cos(\omega t)]$	$\frac{s^2}{(s^2+\omega^2)^2}$

TABLE 2–2 Properties of Laplace Transforms

1	$\mathscr{L}[Af(t)] = AF(s)$
2	$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathscr{L}_{\pm}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathscr{L}_{\pm}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathscr{L}_{\pm}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} \overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathscr{L}_{\pm}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s}$
7	$\mathscr{L}_{\pm}\left[\iint f(t)\,dt\,dt\right] = \frac{F(s)}{s^2} + \frac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s^2} + \frac{\left[\iint f(t)\,dt\,dt\right]_{t=0\pm}}{s}$
8	$\mathscr{L}_{\pm}\left[\int \cdots \int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}}\left[\int \cdots \int f(t)(dt)^k\right]_{t=0\pm}$
9	$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
10	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s) \qquad \text{if } \int_0^\infty f(t)\,dt \text{ exists}$
11	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$
12	$\mathscr{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$
13	$\mathscr{L}[tf(t)] = -\frac{dF(s)}{ds}$
14	$\mathscr{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
15	$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$
16	$\mathscr{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds \qquad \text{if } \lim_{t\to 0}\frac{1}{t}f(t) \text{ exists}$
17	$\mathscr{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$

12- Quadratic Equation

$$As^2 + Bs + C = 0$$

$$s_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

13- Newton's Second Law

Rectlinear motion in the x − direction

$$\sum f_x = m\ddot{x}$$

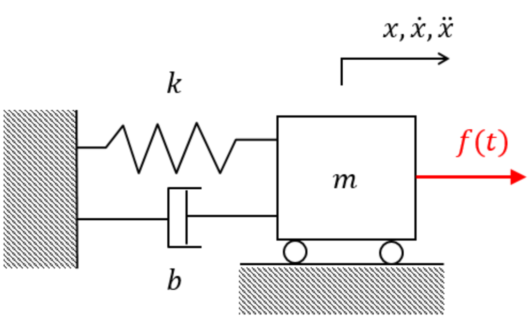
Rotation of a rigid body about pinned point O

$$\sum m_O = J_O\ddot{\theta}$$

14. Initial Value Theorem and Final Value Theorem

$$\lim_{t \to 0}(x(t)) = \lim_{s \to \infty}(sX(s)) \qquad \lim_{t \to \infty}(x(t)) = \lim_{s \to 0}(sX(s))$$

15. Equation of Motion and Transfer Function for a Damped Harmonic Oscillator



$$\sum f_x = m\ddot{x}$$

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

17. Definition of Natural Frequency, Damping Ratio, and Damped Natural Frequency

Given a second order differential equation of motion $A\ddot{x}+B\dot{x}+Cx=g(t)$ where A, B and C are constants, $x(t)$ is the dependent variable and $x(t)$ is the independent variable, the natural frequency, damping ratio and damped natural frequency can be defined.

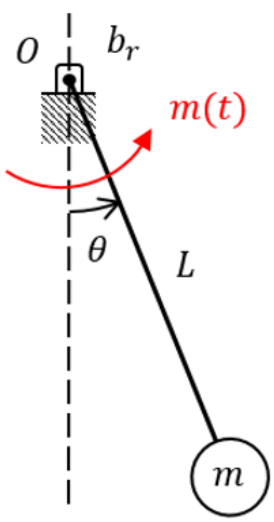
$$\omega_n=\sqrt{\frac{C}{A}}$$

$$\zeta=\frac{B}{2\sqrt{AC}}=\frac{B}{2A\omega_n}$$

$$\omega_d=\omega_n\sqrt{1-\zeta^2}$$

$\zeta=0$	Undamped, Oscillates Indefinitely, Purely Imaginary Roots
$\zeta<1$	Underdamped, Exponentially Decaying Oscillation, Complex Roots
$\zeta=1$	Critically Damped, No Oscillation, Repeated Real Roots
$\zeta>1$	Overdamped, No Oscillation, Distinct Real Roots

18. Simple pendulum with rotational damping and input moment



Nonlinear Equation of Motion

$$mL^2\ddot{\theta}+b_r\dot{\theta}+mgL\sin\theta=m(t)$$

Linearized Equation of Motion

$$mL^2\ddot{\theta}+b_r\dot{\theta}+mgL\theta=m(t)$$

Transfer Function

$$\frac{\Theta(s)}{M(s)}=\frac{1}{mL^2s^2+b_rs+mgL}$$

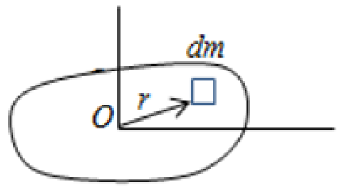
Natural Frequency and Damping Ratio

$$\omega_n=\sqrt{\frac{g}{L}}$$

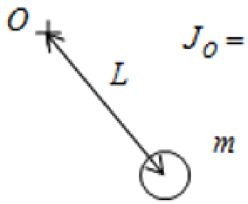
$$\zeta=\frac{b_r}{2mL^2\omega_n}$$

19. Mass moments of inertia

$$J_O = \int_V r^2 dm$$

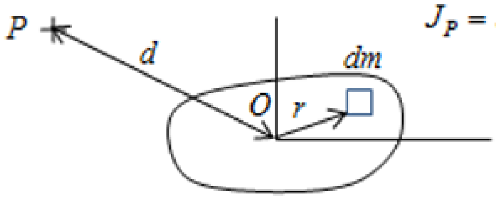


Point Mass



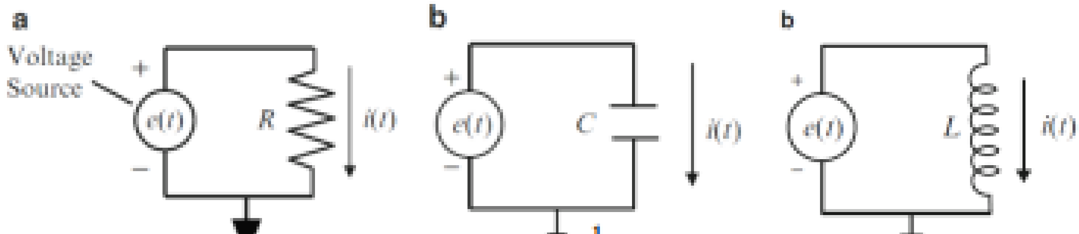
$$J_O = mL^2$$

Parallel axis theorem



$$J_P = J_O + md^2$$

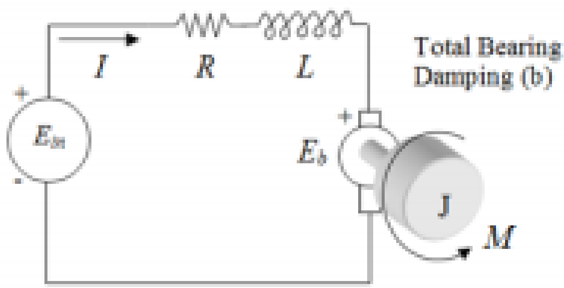
Impedances of passive electrical circuit elements



$Z_R(s) = R$ $\qquad Z_C(s) = \dfrac{1}{Cs}$ $\qquad Z_L(s) = Ls$

Transfer function of a permanent magnet DC-motor (Unloaded)

$$M = KI \qquad (1) \qquad\qquad E_b = K_b\Omega \qquad (2)$$



$$(Js + b)\Omega(s) = M \qquad (4)$$

$$E_{in}(s) - IR - ILs - E_b(s) = 0 \qquad (3)$$

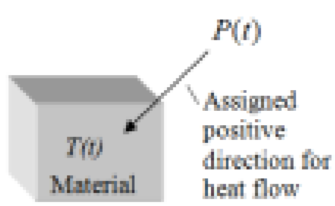
$$\boxed{\frac{\Omega(s)}{E_{in}(s)} = \frac{K}{JLs^2 + (JR + b_r L)s + b_r R + KK_b}}$$

$$\boxed{\tau_{elec} \equiv \frac{L}{R} \qquad \tau_{mech} \equiv \frac{RJ}{KK_b}}$$

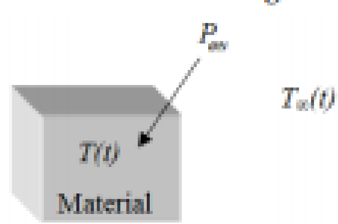
Loaded permanent magnet DC-motor (steady state with load M_l) – Power and Torque Curves

$$M_l = \frac{K}{R}V_0 - \left(\frac{bR + KK_b}{R}\right)\omega_{ss} \qquad\qquad P = M_l\omega_{ss} = \frac{KV_0}{R}\omega_{ss} - \frac{(KK_b + bR)}{R}\omega_{ss}^{\ 2}$$

Thermal Element **Environmental Heat Exchange**



$$mc\dot{T} = P(t)$$



$$P_{env} = hA(T_\infty(t) - T(t))$$

$$\boxed{\tau_{thermal} = \frac{mc}{hA}}$$