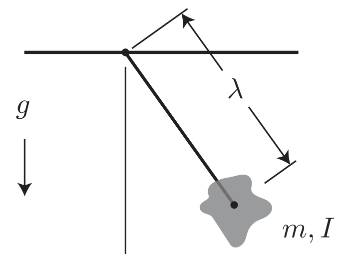
Each of the following five questions is worth 20 points. A passing grade on the exam overall is 70 points. You may use a calculator and attached equation sheet but no other electronic device and no additional reference material during the exam. If you do not understand something, make reasonable assumptions and state them clearly. Note that a table of common Laplace transforms is provided after the final question. Good luck!

1. Consider a planar pendulum comprising a massless rigid rod and an irregularly shaped bob as shown:



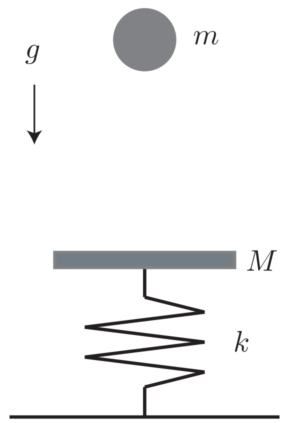
The pendulum pivots frictionlessly about a fixed point on the ceiling. The bob is fixed rigidly to the rod, which connects the pivot point to the bob's center of mass. Let m denote the bob's mass and I its rotational inertia with respect to its center of mass.

Suppose that the pendulum has been released from rest, and oscillates freely under the influence of gravity.

For a fixed value of m, how does the period of oscillation vary with I? If m and I are fixed, how does the period of oscillation depend on the amplitude of oscillation? In terms of the four constants named in the figure, what value does the period approach as the amplitude of oscillation approaches zero? If the pivot point were free to slide laterally along the ceiling, how would this alter the period of oscillation?

Justify your responses mathematically.

2. A ball with mass m is released from rest above a platform with mass M that rests in equilibrium atop a spring with stiffness k, as shown:



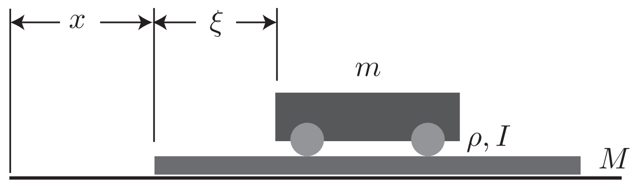
The ball drops a distance h before making contact with the platform, with which it collides with coefficient of restitution e. The system is aligned so that the ball and plate move only vertically as a result of the collision.

In terms of h, e, and the constants shown in the figure, how high will the ball bounce and how far will the plate drop relative to their positions at the moment of contact, assuming that no second collision occurs before each reaches the extreme of its response to the first collision?

Let σ denote the ball's vertical speed immediately after colliding with the plate. (You may have derived a formula for σ, but let's call it σ in case your formula is incorrect.) In terms of σ and the constants shown in the figure, what is the smallest value of k such that the second collision of the ball and plate will occur in the same location?

Justify your responses mathematically.

3. A plate rests on the floor and a two-wheeled cart rests atop the plate as shown:



The plate can slide frictionlessly along the floor. The cart's wheels make slip-free contact with the plate. Let M denote the mass of the plate, m the mass of the cart including both wheels, ρ the radius of each wheel, and I the rotational inertial of each wheel with respect to its center. Let ξ denote the displacement of the cart with respect to a point fixed on the plate and let x denote the displacement of the plate with respect to a point fixed on the floor.

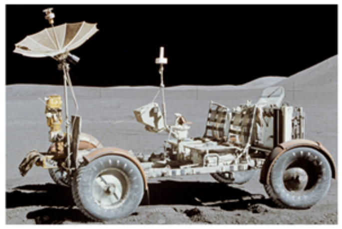
Suppose that one of the cart's wheels rotates freely and frictionlessly while the other is driven by a motor inside the cart. If the system is initially at rest, and then the driven wheel is induced to rotate at a constant rate so that ξ increases by an amount λ over a certain period of time, what is the net change in x over the same period of time? If the cart ceases to move relative to the plate at the end of this increase in ξ, does the plate cease simultaneously to move relative to the floor?

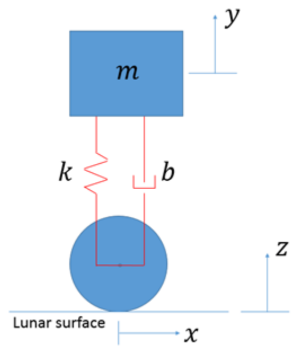
Now suppose that both of the cart's wheels rotate freely and frictionlessly. If the system is initially at rest, and then an external force is applied directly to the plate so that x increases steadily by an amount Λ over a certain period of time, what is the net change in ξ over the same period of time? If the plate ceases to move relative to the floor at the end of this increase in x, does the cart cease simultaneously to move relative to the plate?

Now reconsider the second scenario above, but with viscous friction resisting rotation of the cart's wheels, so that the cart experiences a resistive force proportional to $\dot{\xi}$ when it moves relative to the plate. If the plate is displaced a distance Λ and then halted as above, what is the asymptotic behavior of ξ as a function of time?

Ignore the possibility of the cart's falling off the plate. Justify your answers mathematically.

4. Lunar rover suspension model. Consider the suspension system below, which consists of a rigid wheel (clearly, there are tires in the picture to the left, but we are approximating them to be rigid), spring with spring constant k, and damper with damping coefficient b. z represents the lunar surface height, y represents the vertical displacement of the rover's sprung mass relative to its loaded equilibrium (i.e., when $z = 0$ and only gravitational loading is applied to the rover, it settles to a vertical position of $y = 0$), and x represents the horizontal position of the rover along the moon's surface. All displacements are measured in meters.

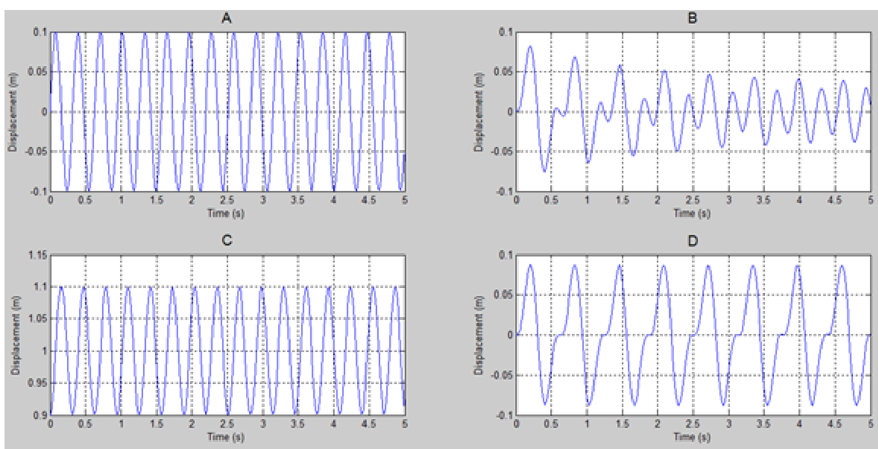




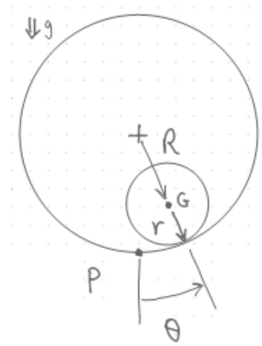
   (a) Given $k = 40000$ N/m, $b = 400$ Ns/m, $m = 400$ kg, and $g = 1.6\ m/s^2$, derive the transfer function from z to y (i.e., derive $(Y(s))/(Z(s))$).

   (b) Assess the stability and DC (steady-state) gain for the lunar rover suspension system.

   (c) Assume that the lunar surface height profile is given by $z = 0.1\sin(x)$ and that the rover is driven at a speed of 20 m/s. Furthermore, assume that $x(0) = 0$, $y(0) = 0$ and $\dot{y}(0) = 0$. Determine which of the following four graphs represents the response of y as a function of time.



5. A uniform disk rolls within a larger cylinder that is fixed in space, as depicted. The rolling disk has radius r, and the larger cylinder has radius $r + R$, such that the center of mass of the disk is at a distance R from the center of the cylinder. The angle θ from the lowest position defines the position of the center of the disk within the cylinder. The mass moment of inertia of the disk about its center is given as $I_G = \frac{1}{2}mr^2$.



(a) If the disk is at $\theta = 0$ and the center of mass has a velocity v, and rolls without slip, determine the expression for v, such that the maximum height of the disk is $\theta = \pi/2$.

(b) If the maximum angle that the disk can reach without slipping is $\theta_{\max}$, determine the static coefficient of friction.