Name: ______________________________

Question:	1	2	3	4	5	Total
Points:	25	25	25	25	25	100
Score:						

Instructions

- Answer any four out of the five questions.
- If you answer more than four, identify the questions to be graded. Otherwise, the first four will be graded.
- Show all the work.
- Please return the questions sheets along with your answer sheets.

1. (25 points) (a) The Fourier series of a function $f(x)$ over the interval $-\pi \leq x \leq \pi$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx.)$$

   Derive the expressions for a_0 and a_n, $n = 1, 2, 3, \ldots$.

   (b) Show that the coefficients a_n, $n = 0, 1, 2, \ldots$ are zero when $f(x)$ is odd.

   (c) Given that $f(x)$ is

$$f(x) = 3 \sin 5x + 7 \cos 8x$$

   over the interval $-\pi \leq x \leq \pi$, what is the Fourier series of $f(x)$?

   (d) If $f(x)$ is discontinuous at $x = 0$, what value does the Fourier series of $f(x)$ converge to at $x = 0$?

2. (25 points) Consider the wave equation

$$C_b^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad , 0 < x < L, \quad t > 0$$

   with the boundary conditions

$$u(0, t) = 0 \text{ and } u(L, t) = 0, \quad t > 0$$

   and the initial conditions

$$u(x, 0) = \alpha \sin \frac{4\pi x}{L} \text{ and } \frac{\partial u}{\partial t}(x, 0) = \beta \sin \frac{4\pi x}{L}$$

   where α and β are some constants.

3. (25 points) Consider the following ordinary differential equation

$$\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-3x}$$

   (a) Provide the homogeneous solution.

   (b) Find the particular solution using the method of undetermined coefficients.

   (c) Use the boundary conditions $y(0) = 5$ and $y'(0) = 0$ to solve for remaining constants and obtain the complete solution.

   (d) Now determine the particular solution of the ODE shown below using the method of variation of parameters.

$$\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y = \sin x$$

   Hint: This ODE has the same homogeneous solution as part a). The particular solution will be of the form $y_p = u_1 y_1 + u_2 y_2$ where y_1 and y_2 are the two linearly independent solutions from part a).

4. (25 points) Consider the following first order ordinary differential equation (ODE) for y(x)

$$3y^2x\frac{dy}{dx} - 3y^3 - \frac{20}{x^2} = 0$$

   (a) Show that the above ODE is not exact by checking the exactness condition.

   (b) Transform the ODE into an exact ODE using an integrating factor.

   (c) Solve for the implicit general solution of the exact ODE from part b).

   (d) Solve the explicit particular solution with the initial condition $y(1) = -2$

5. (25 points) Consider the following second order ODE

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

   (a) Identify the singular points of this equation, if any.

   (b) Derive a recurrence formula (i.e. $c_k = (...)$) for the power series solution around $x = 0$ for the differential equation.

   (c) Express the coefficients $c_2, c_3, c_4, c_5, c_6, c_7$ as a function of c_0 and c_1.

   (d) Use the recurrence formula that was found in (b) to find analytical expressions for the first and the second power series solution that satisfies the differential equation.

   (e) Combine the two solutions in (d) to obtain a general solution for the ODE and determine the unknown constants for initial conditions $y(0) = 2$ and $y'(0) = -1$.