Numerical Methods Equation Sheet

1. Roots of Equations

- Bisection Method

$$x_{\text{mid}} = \frac{x_{\text{lower}} + x_{\text{upper}}}{2}$$

- Newton-Raphson Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- Secant Method

$$x_{i+1} = x_i - f(x_i)\left(\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}\right)$$

2. Linear Algebraic Equations

- Gauss Elimination Method Forward elimination to create an upper triangular matrix, then back substitution to solve for the variables.

- LU Decomposition

$$\mathbf{A} = \mathbf{L}\mathbf{U}, \tag{1}$$

$$\mathbf{L}\mathbf{y} = \mathbf{b} \tag{2}$$

$$\mathbf{U}\mathbf{x} = \mathbf{y} \tag{3}$$

  where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{U}$ is an upper triangular matrix.

- Cholesky Factorization

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T$$

  where $\mathbf{L}$ is a lower triangular matrix with positive diagonal elements and $\mathbf{L}^T$ is the transpose of $\mathbf{L}$.

- Gauss-Seidel Iteration

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

- Jacobi Iteration

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j^{(k)}\right)$$

- Successive Over-Relaxation (SOR)

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

where ω is the relaxation factor, and $0 < \omega < 2$.

3. Curve Fitting and Interpolation

- Lagrange Interpolating Polynomial

$$P(x) = \sum_{i=0}^{n} y_i \prod_{\substack{0\le j\le n \\ j\ne i}} \frac{x - x_j}{x_i - x_j}$$

- Newtons Divided Difference Interpolation

$$P(x) = f[x_0] + f[x_0,x_1](x-x_0) + \ldots + f[x_0,x_1,\ldots,x_n](x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

- Least Squares Regression (Linear)

$$y = a_0 + a_1 x$$

$$a_1 = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}, \quad a_0 = \frac{\sum y - a_1 \sum x}{n}$$

4. Numerical Differentiation and Integration

- Finite Difference Approximation
  - Forward Difference (first and second derivatives)

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$f''(x) \approx \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

  - Backward Difference

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

$$f''(x) \approx \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}$$

  - Central Difference

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

- Numerical Integration
  - Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}[f(a)+f(b)]$$

  - Simpsons 1/3 Rule: For an even number of intervals (n = 2, 4, 6, ...):

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

  - Simpsons 3/8 Rule: For a multiple of three intervals (n = 3, 6, 9, ...):

$$\int_a^b f(x)\,dx \approx \frac{3(b-a)}{8}\left(f(a)+3f\left(a+\frac{b-a}{3}\right)+3f\left(a+\frac{2(b-a)}{3}\right)+f(b)\right)$$

  - Gauss-Legendre Quadrature Rules: The integral approximation is given by:

$$\int_{-1}^1 f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

    where x_i are the Gauss-Legendre nodes and w_i are the corresponding weights. Below are the values of nodes and weights for different orders of Gauss-Legendre Quadrature Rules:

    1. 1st Order (n=1)

$$x_1 = 0, \quad w_1 = 2$$

    2. 2nd Order (n=2)

$$x_{1,2} = \pm\frac{1}{\sqrt{3}}, \quad w_1 = w_2 = 1$$

    3. 3rd Order (n=3)

$$x_1 = 0, \quad x_{2,3} = \pm\sqrt{\frac{3}{5}}, \quad w_1 = \frac{8}{9}, \quad w_2 = w_3 = \frac{5}{9}$$

    4. 4th Order (n=4)

$$x_{1,2} = \pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}, \quad x_{3,4} = \pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$$

$$w_{1,4} = \frac{18+\sqrt{30}}{36}, \quad w_{2,3} = \frac{18-\sqrt{30}}{36}$$

5. Solving Ordinary Differential Equations (ODEs)

- Eulers Method

$$y_{i+1} = y_i + hf(x_i, y_i)$$

- Runge-Kutta Method (Fourth Order)

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where:

$$\begin{aligned}
k_1 &= f(x_i, y_i), \\
k_2 &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right) \\
k_3 &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right), \\
k_4 &= f(x_i + h, y_i + hk_3)
\end{aligned}$$

6. Solving Partial Differential Equations (PDEs)

- Finite Difference Method for Heat Equation

$$u_i^{n+1} = u_i^n + \alpha \frac{\Delta t}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

where α is the thermal diffusivity.

7. Optimization

- Newtons Method for Optimization

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)}$$

8. Norms

- Vector Norms
  - 1-Norm (Taxicab or Manhattan Norm)

    $$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

    where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ is a vector in $\mathbb{R}^n$.
  - 2-Norm (Euclidean Norm)

    $$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

  - Infinity Norm (Maximum Norm)

    $$\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

- Matrix Norms
  - 1-Norm (Maximum Absolute Column Sum)

    $$\|\mathbf{A}\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|$$

    where $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix.

- Frobenius Norm

$$\|\mathbf{A}\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

- Infinity Norm (Maximum Absolute Row Sum)

$$\|\mathbf{A}\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

- Matrix 2-Norm (Spectral Norm) The matrix 2-norm (also known as the spectral norm) is the largest singular value of $\mathbf{A}$:

$$\|\mathbf{A}\|_2 = \sigma_{\text{max}}$$

  where σ_{max} is the largest singular value of $\mathbf{A}$.