Numerical Methods
Ph.D. Qualifying Exam, Fall 2025

The duration of this exam is 3 hours. There are 5 questions and the students must work on 4 of them. If more than 4 questions are attempted, the student must specify which are the 4 to be graded. If not specified, the first 4 problems will be graded.

Student Name:

Problem	Max Points	Student's grade
1	25	
2	25	
3	25	
4	25	
5	25	
Exam Total		

Problem 1

You are given evenly spaced data for a smooth function $f(x)$ on $[0, 0.6]$ with step $h = 0.1$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	1.0000	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488

(a) Using the first four points, construct the cubic Newton interpolant and evaluate $p_3(0.25)$. Show the divided-difference table.

(b) With $h = 0.1$, use forward difference method to approximate $f'(0.3)$.

(c) Approximate $\int_0^{0.6} f(x)\,dx$ using composite Simpson's rule with panel width $h = 0.2$.

Problem 2

We want to solve the nonlinear system

$$F(\mathbf{x}) = \begin{bmatrix} x_1^2 + x_2^2 - 4 \\ e^{x_1} + x_2 - 1 \end{bmatrix} = \mathbf{0}.$$

(a) Starting from $\mathbf{x}^{(0)} = (1, 0)^\top$, perform two iterations of Newton–Raphson for systems:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - J(\mathbf{x}^{(k)})^{-1} F(\mathbf{x}^{(k)}),$$

where J is the Jacobian. Show $J(\mathbf{x}^{(0)})$, $F(\mathbf{x}^{(0)})$, $\mathbf{x}^{(1)}$, and compute $\mathbf{x}^{(2)}$ approximately. Comment on expected convergence.

(b) Consider the matrix

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

Show that A is symmetric positive definite (SPD). What does SPD guarantee about uniqueness of solutions?

(c) Find the Cholesky factorization $A = LL^\top$ and solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (1, 2, 3)^\top$.

Problem 3

Part I: Newton–Raphson for Systems

Consider the nonlinear system

$$\begin{cases} x_1^2 + x_1 x_2 - 10 = 0, \\ x_2 + 3x_1 x_2^2 - 57 = 0. \end{cases}$$

(a) Starting from $\mathbf{x}^{(0)} = (1,3)^\top$, compute the Jacobian J, evaluate $F(\mathbf{x}^{(0)})$, and perform one Newton step to obtain $\mathbf{x}^{(1)}$.

(b) Does the Newton-Raphson method always converge? If not, briefly discuss when divergence may occur.

Part II: ODE IVP

Solve the IVP

$$y'(t) = -2y(t) + t^2, \quad y(0) = 1,$$

on $t \in [0, 0.2]$.

(c) Using the classical RK4 method with step $h = 0.2$, compute $y(0.2)$. Show the stage values $k_1, \dots, k_4$.

Problem 4

A simple recursive filter is often used in signals analysis for which data x_i is sampled at time intervals Δt and smoothed to provide the new signal s_i using the equation

$$s_i = \alpha x_i + (1-\alpha) s_{i-1}. \tag{1}$$

This is algorithm is called exponential smoothing.

a) For a unit step $x(t) = \mathit{const.} = 1$ and a value $\alpha = 0.5$, determine the first six output values of s for $\Delta t = 1$.

b) Compare these results with a function of the form

$$s = 1 - e^{-t/\tau},$$

where

$$\tau = \frac{\Delta t}{\ln(1-\alpha)},$$

over the same range of time.

c) Compare and comment on these results.

Problem 5

State the relative merits of LU factorization for solving linear equations of the form

$$[A]\{x\} = \{b\}.$$

a) Determine L and U for the matrix

$$[A] = \begin{bmatrix} 3 & -4 & 4 \\ 1.5 & 3.5 & -2 \\ 1 & -1 & 4 \end{bmatrix}.$$

b) For

$$\{b\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \tag{2}$$

solve for the three values of x.